DEDEKIND'S CUTS AND REAL NUMBERS

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The text below is an example of the routine job required when you introduce a new mathematical notion. It consists in motivations, formal definitions and meticulous checks that what we get is free of contradictions and indeed answers our wishes. We arrange them in the form of problems and sometimes supply solutions: you are expected to be able to supply the missing solutions.

1. Construction of real numbers: adding nonexistent solutions to infinite systems of two-sided inequalities

In the same way as we "added" to $\mathbb{Q}$ non-existent solutions of polynomial equations, we can add non-existent solutions to (infinite) systems of consistent two-sided inequalities of the form $l \leqslant x \leqslant r$, $l, r \in \mathbb{Q}$. To do this, we need to reduce all such systems to some canonical form, so that different systems could be compared and manipulated with.

Problem 1. *Why finite systems cannot be used to "add" new numbers?*

Solution. Any finite number of equations $l_k \leqslant r_k$, $k = 1, \dots, n$, can be replaced by a single equivalent inequality $l_* \leqslant x \leqslant r_*$, $l_* = \max_k l_k$, $r_* = \min_k r_k$. If $l_* > r_*$, the system is inconsistent. If $l_* < r_*$, it is satisfied by at least two different rational solutions, hence cannot be considered a "number". If $l_* = r_*$, then this is an "old" rational number, and we don't get anything new.

Any such (finite or infinite) system of equations is defined by two subsets, $L = \{l_k\}$ and $R = \{R_k\}$. For simplicity, we will say about "the system (L, R)". To be self-consistent, we need the condition

$$L \leqslant R, \qquad \text{i.e.,} \quad \forall l \in L, r \in R \quad l \leqslant r$$

(note how we abused the sign $\leqslant$, using it between subsets rather than between numbers!).

Problem 2. *What happens with the system of equations if the condition $L \leqslant R$ is violated?*

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In order for this system to define a "unique" number, we want the infinite system (L, R) to be *maximal*, i.e., extend L and R as much as possible without running into a contradiction. We will assume that $L \cup R = \mathbb{Q}$. Intuitively this means that *any rational number can be compared with the "new number" defined by the infinite system and placed either into L, or into R.*

Definition 1. A *cut* (in full, a Dedkind cut) is a pair (L, R) of two subsets $L, R \subseteq \mathbb{Q}$, such that:

(1) $L, R \neq \varnothing$;
(2) $L \leqslant R$ in the above sense;
(3) $L \cup R = \mathbb{Q}$.

The set of all cuts (pairs as above) we denote by $\mathfrak{D}$.

Problem 3. *Prove that for any cut (L, R) the intersection $L \cap R$ can be either empty, or consist of at most one number $q \in \mathbb{Q}$.*

Solution. If there were two *different* numbers $q_1 < q_2 \in L \cap R$, this would mean that $q_2 \in R$ and $q_1 \in L$ in contradiction with the requirement $L \leqslant R$.

Definition 2. If $L \cap R$ consists of a single number $q \in \mathbb{Q}$, we call this cut *rational* and identify it with the rational number q. Otherwise we call the cut *irrational.*

In this way we have the embedding $\mathbb{Q} \subseteq \mathfrak{D}$.

Remark 1. If $L \cap R = \{q\}$, then the rational number q can be removed from either L or R (but not from both sets, otherwise $L \cup R \neq \mathbb{Q}$) and still be the unique solution of all inequalities of the two new systems. This means that rational numbers admit three slightly different representation by cuts. This is manifested in the "equality" $0.9999999999999 \cdots = 1.0000000000000\ldots$ between infinite decimal fractions, see below.

Remark 2. In order to make things simpler, we will denote rational cuts by the corresponding rational numbers and write $q \in \mathbb{Q}$ instead of (L, R), $L = \{l \leqslant q\}$, $R = \{r \geqslant q\}$. For the same reason we say that all three (formally different) cuts representing the same rational number, are *equal* (instead of "equivalent").

Problem 4. *Show that $\mathbb{Q} \subsetneq \mathfrak{D}$.*

Solution. Let $R = \{r > 0 : r^2 \geqslant 2\}$, $L_+ = \{l > 0 : l^2 \leqslant 2\}$ and $L = \{l \leqslant 0\} \cup L_+$. Prove that this cut is irrational.

Problem 5. *Prove that for any two non-empty subsets $L^\circ \leqslant R^\circ$ in $\mathbb{Q}$, there exist (eventually, more then one) cut $(L, R) \in \mathfrak{D}$, such that $L^\circ \subseteq L$, $R^\circ \subseteq R$.*

Solution. If there exists a rational number q such that $L^\circ \leqslant q \leqslant R^\circ$, then we can use any such number to construct the extension. Otherwise, we can use the cut

$$(L, R), \qquad L = L^\circ - \mathbb{Q}_+, \quad R = R^\circ + \mathbb{Q}_+.$$

Here $\mathbb{Q}_+ = \{q \in \mathbb{Q} : \ q \geqslant 0\}$. Check that this is indeed a legitimate cut!

2. Operations on cuts: arithmetic and order

Using rules of manipulation with inequalities, given two infinite systems $L' \leqslant x \leqslant R'$ and $L'' \leqslant y \leqslant R''$, we can construct infinite system of inequalities for their sum, product e.a.

Yet because of existence of negative numbers and the way how multiplication by negative numbers affects inequalities, we better start with the order.

Definition 3. For two cuts (L', R') and (L'', R'') we write that $(L', R') \leqslant (L'', R'')$, if $L' \subseteq L''$ and, respectively, $R' \supseteq R''$.

Note that we again abused even more the sign $\leqslant$! Now it is allowed to occur between two cuts rather than two subsets!

Problem 6. *Prove that for any two cuts (L', R') and (L'', R'') we have one of the two possibilities:*

1. $(L', R') \leqslant (L'', R'')$;
2. $(L'', R'') \leqslant (L', R')$, *in which case we will write* $(L', R') \geqslant (L'', R'')$.

If both inequalities occur simultaneously, then the two cuts correspond to the same rational number.

Problem 7. *Give the definition of the* strict *inequality $(L', R') < (L'', R'')$ for cuts.*

Solution. We say that $(L', R') < (L'', R'')$, if $(L', R') \leqslant (L'', R'')$ and $(L', R') \neq (L'', R'')$. Beware of Remark 2!

Problem 8. *Show that the strict inequality introduced this way for $\mathfrak{D}$, coincides with the usual strict inequality for $\mathbb{Q} \subset \mathfrak{D}$.*

Problem 9. *Prove that for three* irrational *cuts, one and only one sign $<, =, >$ can be placed between them.*

Problem 10. *Show that* $(L, R) \geqslant 0$ *if and only if* $L_+ = L \cap \mathbb{Q}_+ \neq \varnothing$.

Definition 4. For two cuts (L', R') and (L'', R'') we define by their sum $(L', R') + (L'', R'')$ as the cut $(L' + L'', R' + R'')$.

Problem 11. *How do you understand sign* $+$ *between two subsets of* $\mathbb{Q}$*? Give the definition of the difference of two cuts.*

Problem 12. *Describe the cut* $-(L, R)$*, the difference* $0-(L, R)$*. Prove that either* (L, R) *or* $-(L, R)$ *is always nonnegative.*

Definition 5. For two *nonnegative* cuts $(L', R'), (L'', R'') \geqslant 0$ we define their product $(L', R') \cdot (L'', R'')$ as the cut $(L'_+ \cdot L''_+, R' \cdot R'')$.

Problem 13. *Define the product of any two cuts in the general case.*

Problem 14. *Define the inverse* $1/(L, R)$ *for a nonzero cut and prove that it always exists.*

Problem 15. *Prove that the operations* $\pm, \cdot, /$ *introduced on* $\mathfrak{D}$*, make it into a field.*

We can now conclude that $\mathfrak{D} \supsetneq \mathbb{Q}$ is an extension of the field $\mathbb{Q}$ which is also ordered, and the order coincides with that on $\mathbb{Q}$.

It remains to see why $\mathfrak{D}$ is "better" than $\mathbb{Q}$ with respect to absence of "holes".

3. Completeness of cuts

Our main reason to be unsatisfied with rational numbers is that there are too many "holes" between them. In particular:

- A function continuous on a segment and changing its sign there, may have no rational roots (assuming you know what the continuity is).
- A sequence of rational points which obviously must converge, may have no rational limit (assuming you know what the limit is).
- A sequence of nested non-empty segments may have empty intersection.

For any of these reasons say that the system of rational numbers is incomplete (or topologically incomplete, to be more precise).

To simplify the life, we will address only the last manifestation of incompleteness.

Definition 6. A *nested family of subsets* is any infinite decreasing sequence of non-empty subsets (of $\mathbb{Z}, \mathbb{Q}, \mathfrak{D}, \dots$).

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots, \qquad A_n \neq \varnothing$$

Because of the nested structure, any finite intersection $A_1 \cap \cdots \cap A_n$ is equal to A_n and is by definition non-empty. The key question is about the *infinite* intersection

$$A_* = \bigcap A_n = \bigcap_{n=1}^{\infty} A_n = \{x \colon \forall n \in \mathbb{N}\ x \in A_n\}.$$

Problem 16. *If $A_n \subseteq \mathbb{Z}$ and A_1 is finite, then $A_* \neq \varnothing$. Prove.*

Solution. The number of elements $N_n = \#\{x : x \in A_n\}$ is a finite natural (positive) number. By the nested structure,

$$N_1 \geqslant N_2 \geqslant \cdots \geqslant N_n \geqslant \cdots, \qquad N_n > 0.$$

Obviously, the only possibility is that N_n stabilizes at a certain moment, $N_n = N_{n+1} = \cdots = N_* > 0$.

Problem 17. *Let $A_n = \{k \in \mathbb{Z} : \ k \geqslant n\}$. Prove that $\bigcap A_n = \varnothing$.*

Denote by $]a, b[= \{a < x < b\}$ an open interval (the endpoints not included) and $[a, b] = \{a \leqslant x \leqslant b\}$ the closed segment including the endpoints; this notation makes sense in $\mathbb{Z}, \mathbb{Q}, \mathfrak{D}$.

Problem 18. *For the nested system of open intervals $]0, \frac{1}{n}[$ the infinite intersection is empty, the infinite intersection of closed segmets $[0, \frac{1}{n}]$ is non-empty. Prove it.*

Problem 19. *Construct a nested family of closed segments on $\mathbb{Q}$ with an empty intersection.*

Remark 3. In the above example the sequence of left endpoints must converge, but has no limit in $\mathbb{Q}$. The same about the sequence of right endpoints. If you remember the definition of the limit, prove it.

Theorem 1. *A nested family of* closed *segments in $\mathfrak{D}$ has a non-empty intersection in $\mathfrak{D}$.*

Problem 20. *Prove this theorem assuming that all segments have rational endpoints.*

Solution. Denote by L° (resp, R°) the sets of left (resp., right) endpoints. Show that $L^\circ \leqslant R^\circ$). Show that (L°, R°) can be completed to be a full cut (L, R) (in many ways). Prove that any such cut belongs to the infinite intersection.

Problem 21. *Prove the Theorem in the general case when left and right endpoints are themselves cuts from $\mathfrak{D}$.*

Theorem 2. *Let $A \leqslant B$ two non-empty subsets of $\mathfrak{D}$ (how do you understand this notation?). Prove that there exists a cut $\alpha = (L, R)$ which separates A and B: $A \leqslant \alpha \leqslant B$.*

Problem 22. *Prove this theorem.*

4. Conclusion

We embedded the field of rational numbers $\mathbb{Q}$, which is an ordered field, into a larger ordered field, whose elements can be described by infinite systems of inequalities $L \leqslant x \leqslant R$, $L, R \subseteq \mathbb{Q}$. If such a system has a rational solution, so be it, otherwise it is interpreted as a new, *real* number. The time is ripe to replace $\mathfrak{D}$ by $\mathbb{R}$, the standard notation for real numbers.

Elements of $\mathfrak{D} = \mathbb{R}$ can be encoded by the infinite decimal fractions: each such fraction is an algorithm to produce two infinite sets L°, R° which can be completed to a cut in $\mathfrak{D}$ in an obvious way. It is an easy exercise to derive from this description the rules of manipulation with decimal fractions.

5. Immediate gain: new operations

If $A \subset \mathbb{Q}$ is a non-empty finite set, then it always has a minimum and a maximum, denoted by $\min A$ and $\max A$. This is no longer the case if A is infinite, and can happen for two reasons:

(1) A is unbounded: if $\forall r \in \mathbb{Q}\ \exists x \in A:\ x > r$, then there is no maximum (the same about minimum),
(2) A is bounded, but the maximum is "not achieved": if $A = \{-\frac{1}{n}\} \subseteq \mathbb{Q}$, there is no maximal element in A.

If A is bounded for above, $A \leqslant r$, $r \in \mathbb{Q}$, then A has infinitely many upper bounds, but it may happen that there is no *smallest* upper bounds.

Problem 23. *Give examples of a set $A \subset \mathbb{Q}$ bounded from above, which has a smallest upper bound and which has no smallest upper bound.*

This is impossible in $\mathbb{R}$.

Theorem 3. *If $A \subseteq \mathbb{R}$ is bounded from above, then there exists a smallest upper bound $r_* \in \mathbb{R}$, the number such that*:

(1) *$A \leqslant r_*$,*
(2) *$A \leqslant r$ for any $r \in \mathbb{R}$ implies that $r* \leqslant r$.*

Problem 24. *Prove this theorem.*

Solution. Denote by $L^\circ = A$ and R° be the set of all upper bounds for A in $\mathbb{R}$. Prove that $L^\circ \leqslant R^\circ$. Prove that there is at most one real number $z \in \mathbb{R}$ separating these two sets. Prove that $z \in R^\circ$.

Definition 7. The smallest upper bound is denoted by $\sup A$. In the same way the biggest lower bound $\inf A$ is defined if A is bounded from below.

The operations $\inf, \sup$ are the best substitutes for the operations $\min, \max$ for the case of infinite subsets.